

Graph Theory

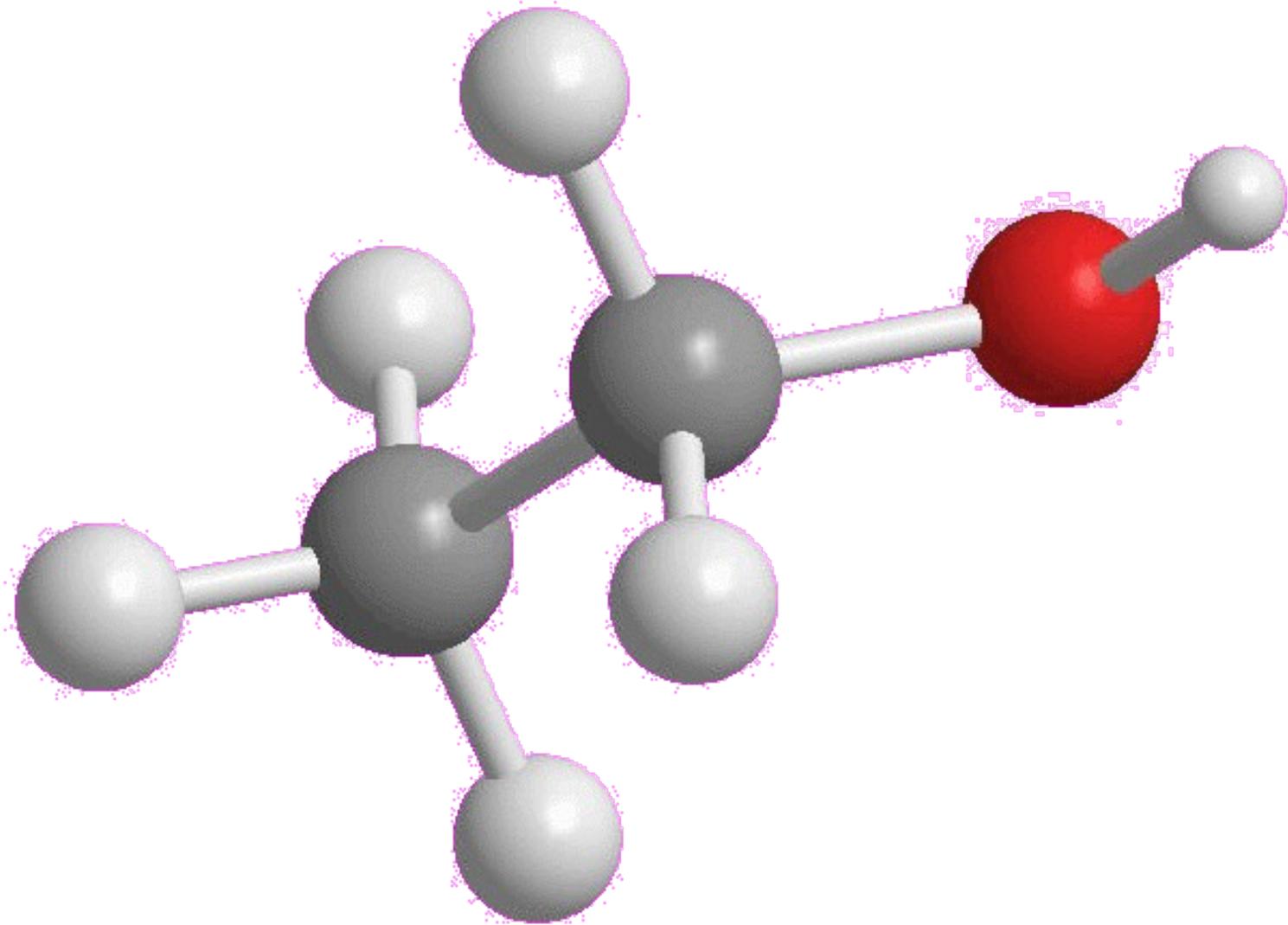
Part One

Outline for Today

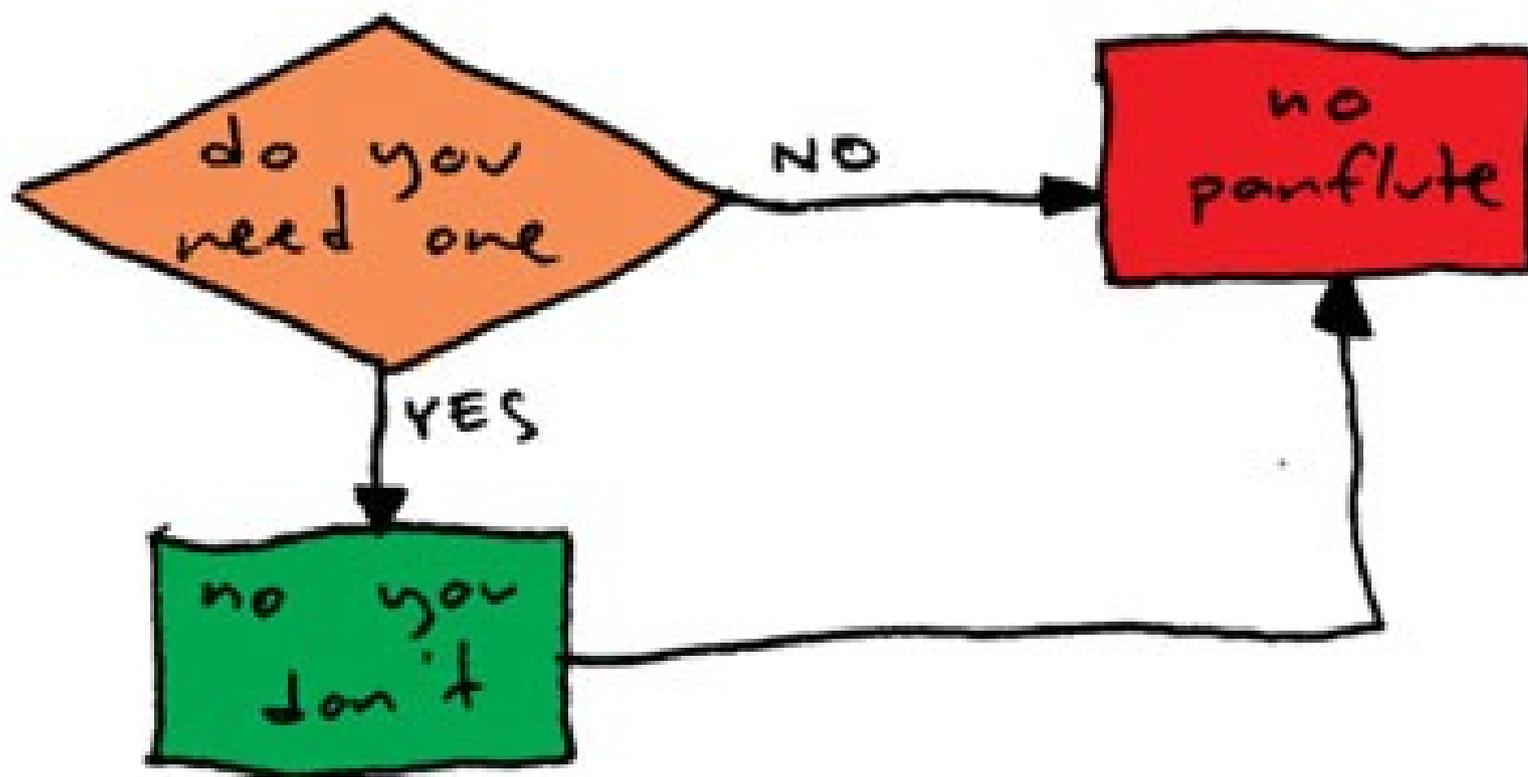
- ***Graphs and Digraphs***
 - Two fundamental mathematical structures.
- ***Independent Sets and Vertex Covers***
 - Two structures in graphs.
- ***Proofs on Graphs***
 - Reprising themes from last week.

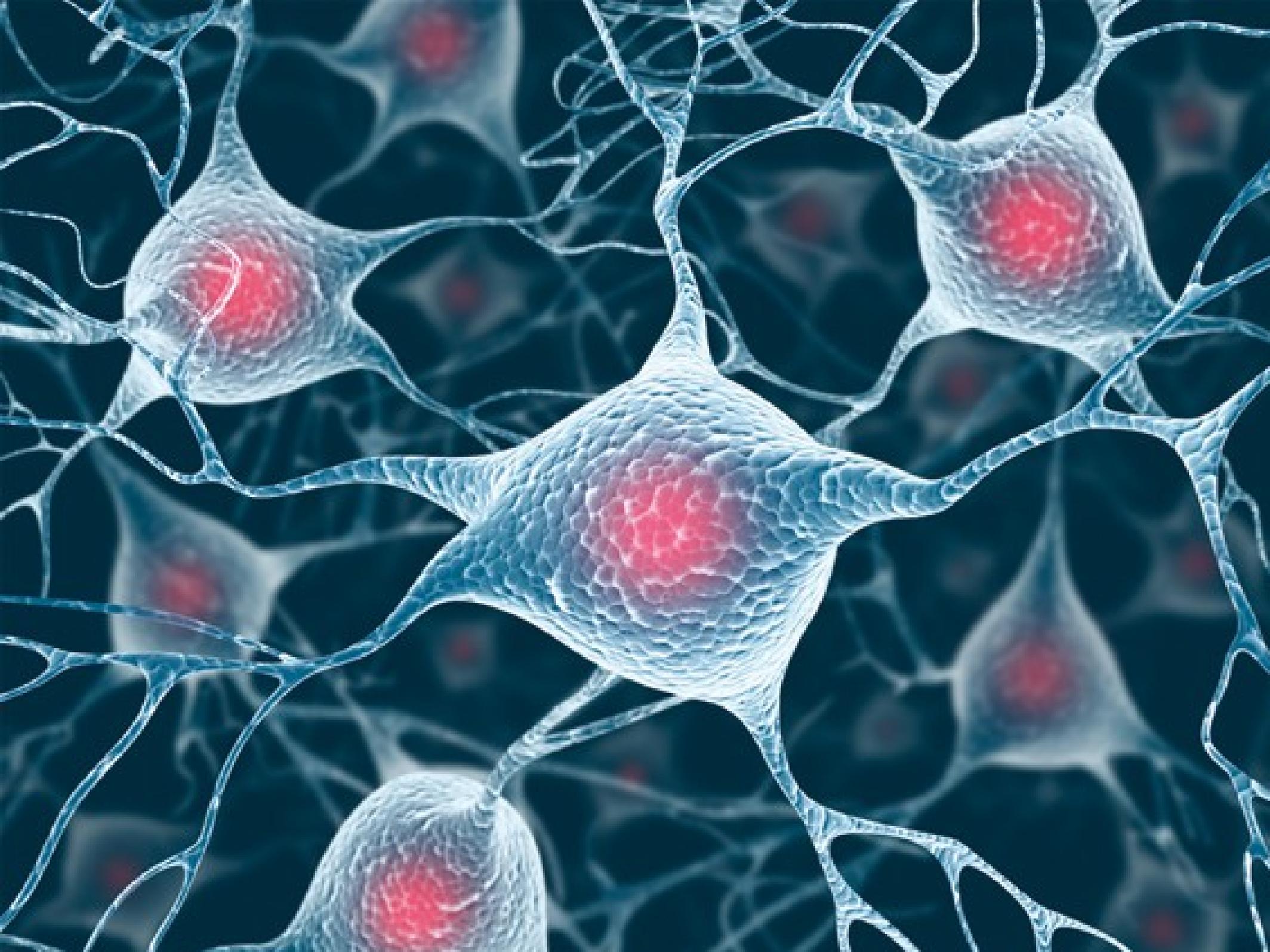
Graphs and Digraphs

Chemical Bonds



PANFLUTE FLOWCHART







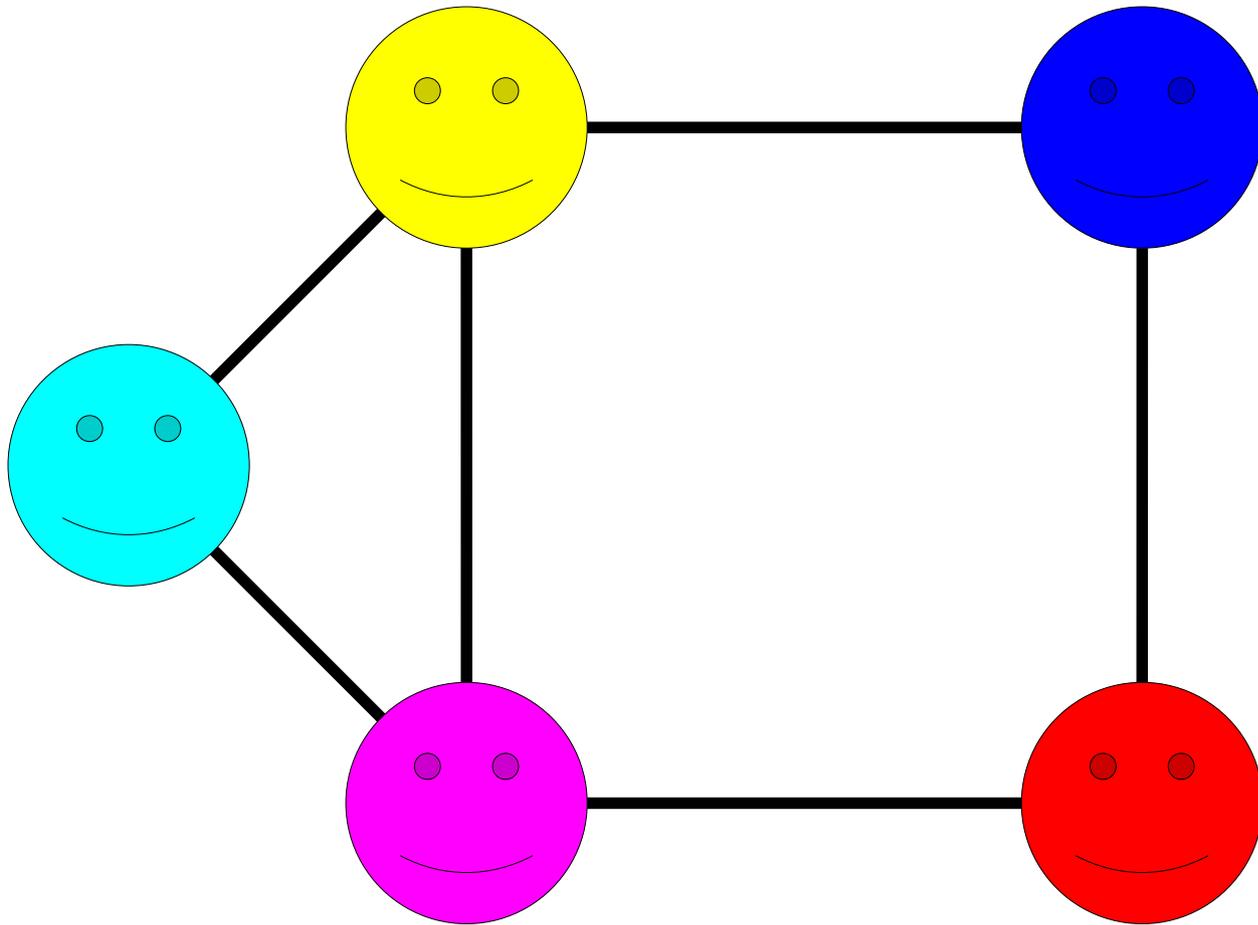
WeChat



What's in Common

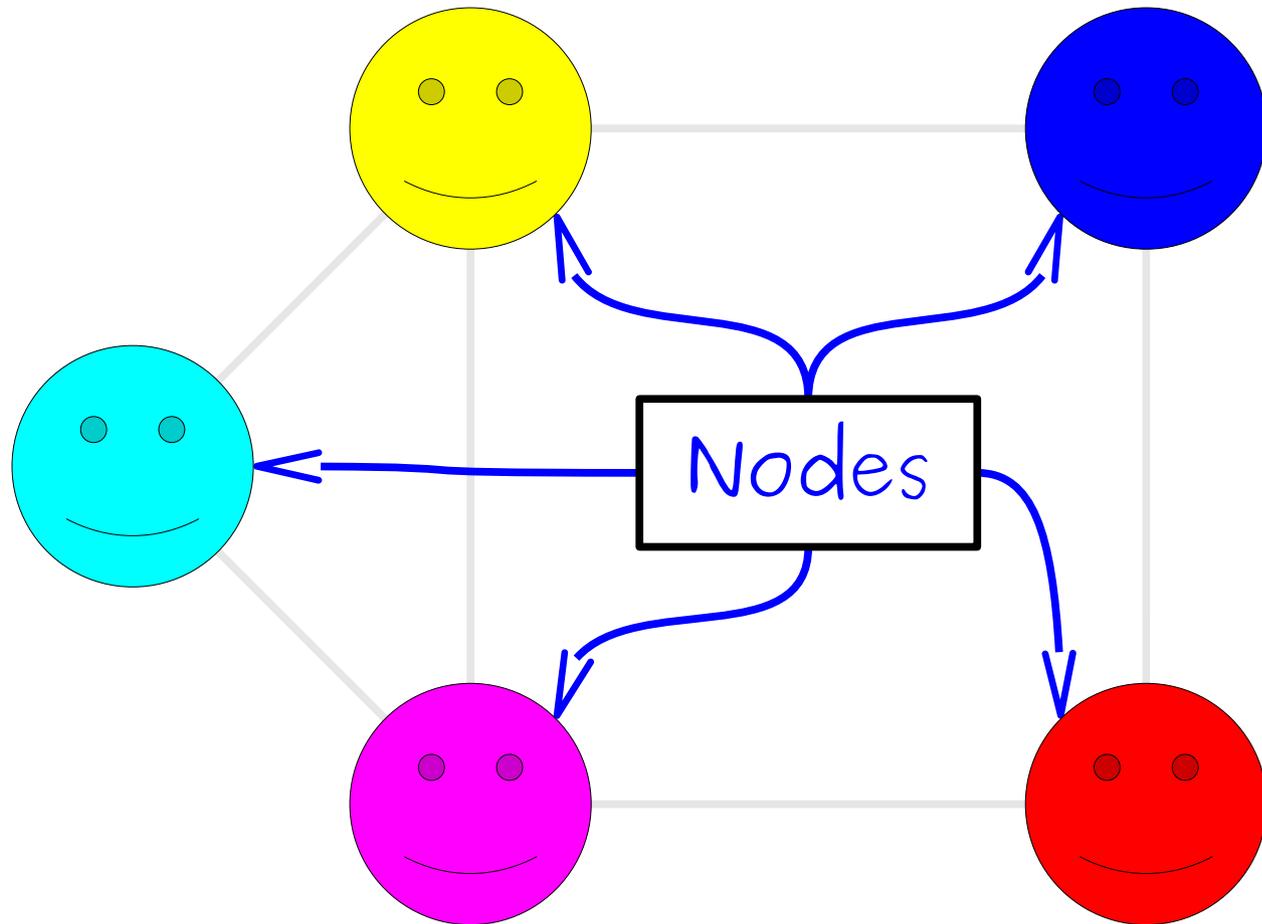
- Each of these structures consists of
 - a collection of objects and
 - links between those objects.
- **Goal:** find a general framework for describing these objects and their properties.

A **graph** is a mathematical structure for representing relationships.



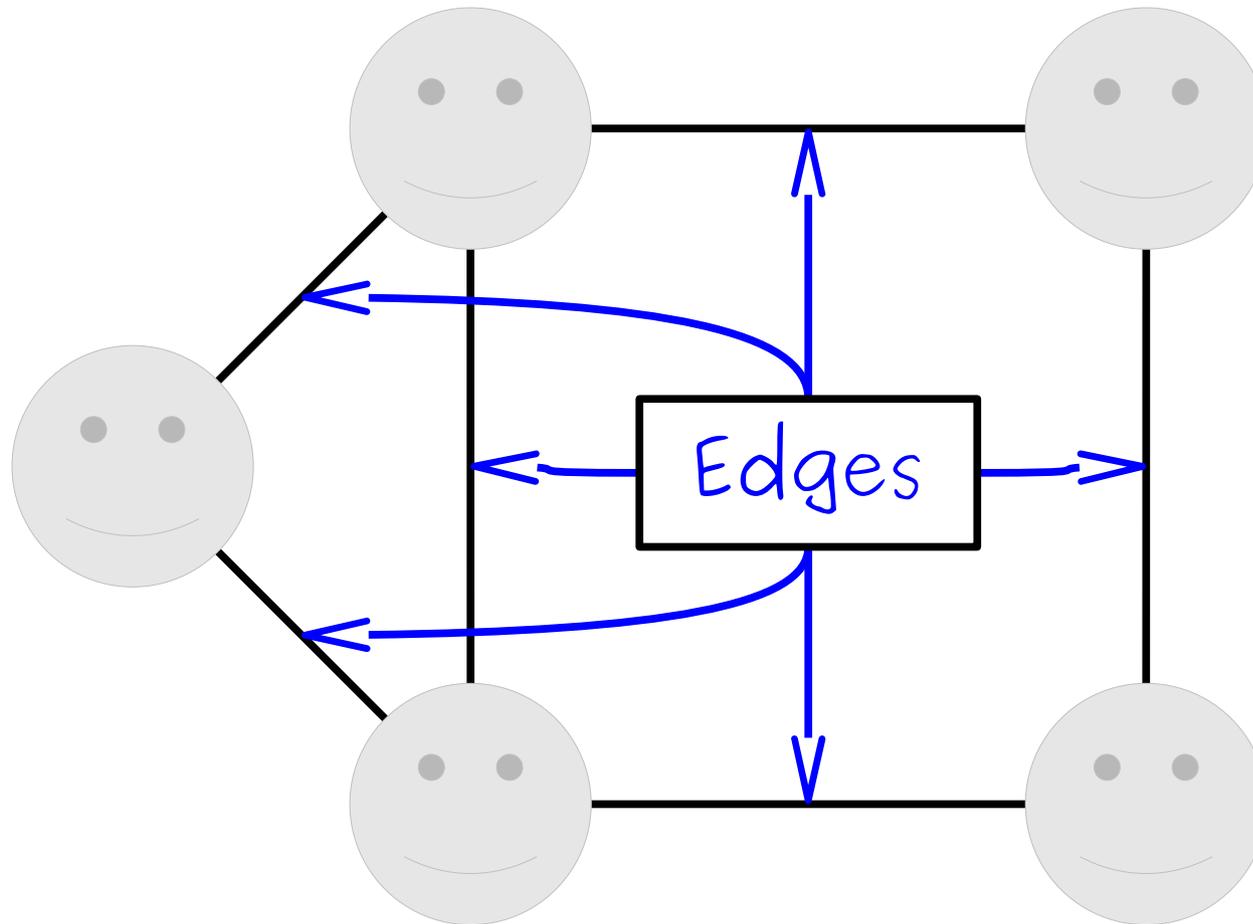
A graph consists of a set of **nodes** (or **vertices**) connected by **edges** (or **arcs**)

A **graph** is a mathematical structure for representing relationships.



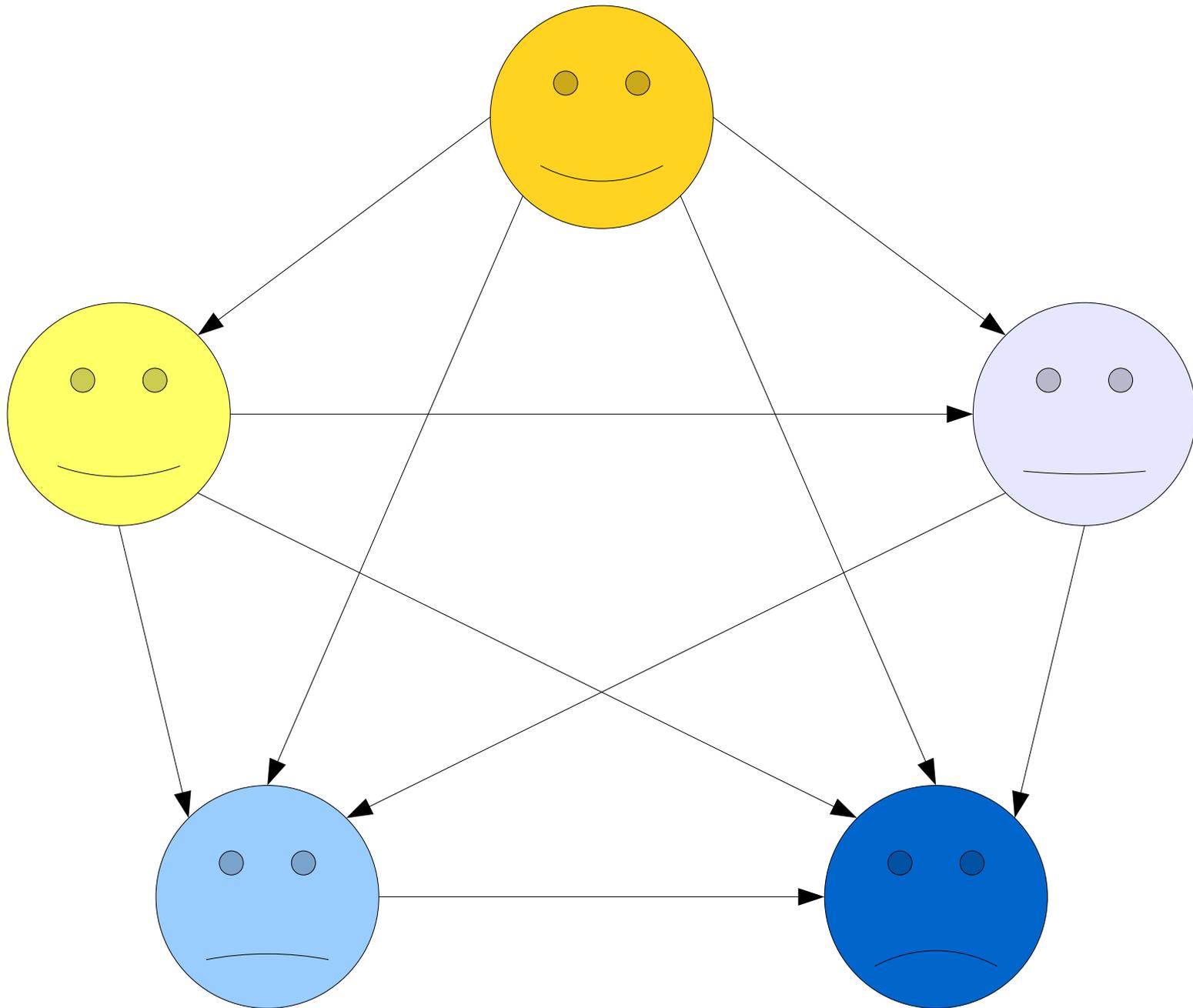
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A **graph** is a mathematical structure for representing relationships.

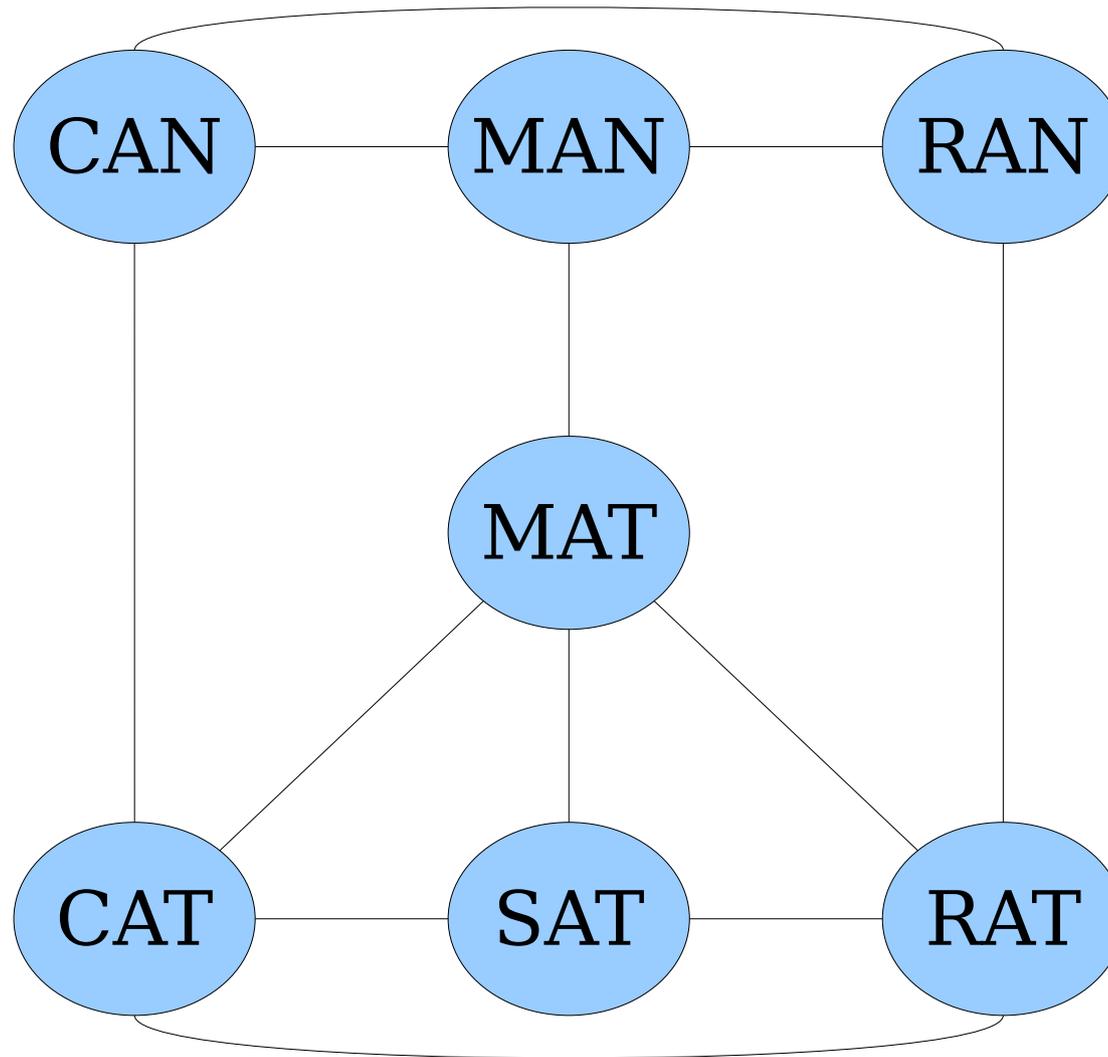


A graph consists of a set of **nodes** (or **vertices**) connected by **edges** (or **arcs**)

Some graphs are *directed*.



Some graphs are *undirected*.



Graphs and Digraphs

- An ***undirected graph*** is one where edges link nodes, with no endpoint preferred over the other.
- A ***directed graph*** (or ***digraph***) is one where edges have an associated direction.
 - (There's something called a ***mixed graph*** that allows for both types of edges, but they're fairly uncommon and we won't talk about them.)
- Unless specified otherwise:
 - 👉 ***“Graph” means “undirected graph”*** 👈

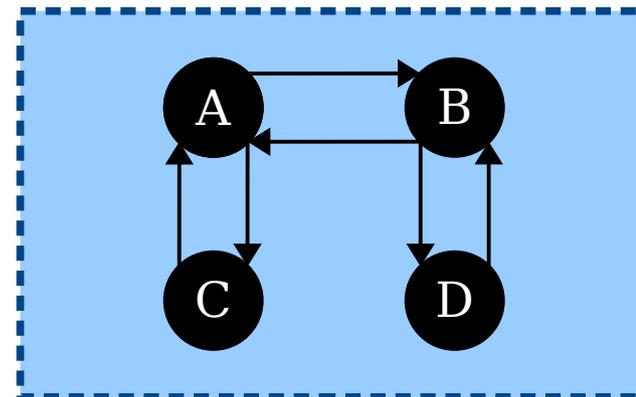
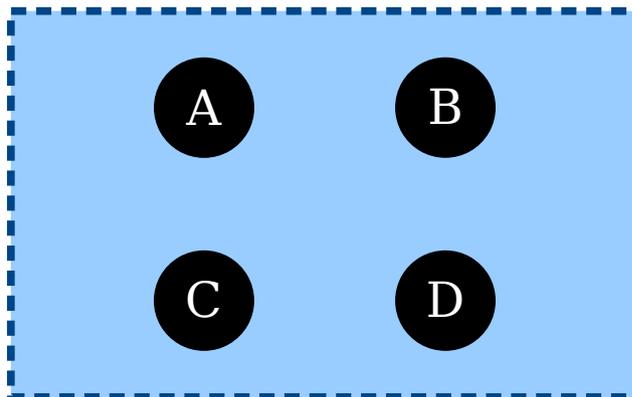
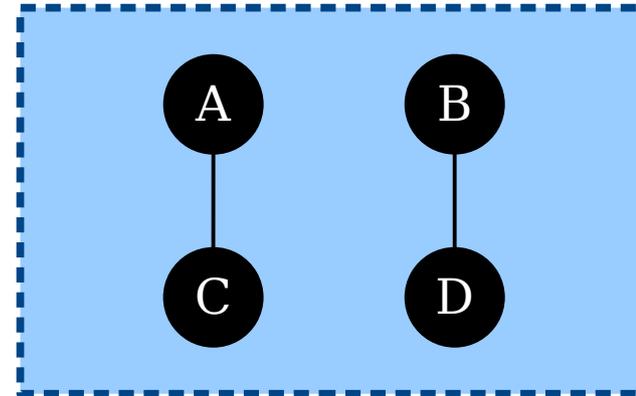
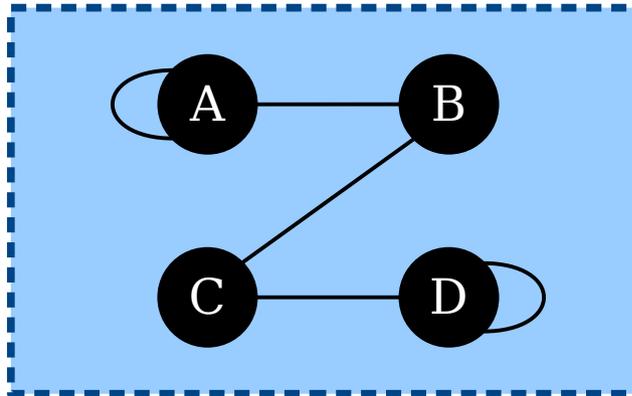
Formalizing Graphs

- How might we define a graph mathematically?
- We need to specify
 - what the nodes in the graph are, and
 - which edges are in the graph.
- The nodes can be pretty much anything.
- What about the edges?

Formalizing Graphs

- An **unordered pair** is a set $\{a, b\}$ of two elements $a \neq b$. (Remember that sets are unordered.)
 - For example, $\{0, 1\} = \{1, 0\}$
- An **undirected graph** is an ordered pair $G = (V, E)$, where
 - V is a set of nodes, which can be anything, and
 - E is a set of edges, which are *unordered* pairs of nodes drawn from V .
- A **directed graph** (or **digraph**) is an ordered pair $G = (V, E)$, where
 - V is a set of nodes, which can be anything, and
 - E is a set of edges, which are *ordered* pairs of nodes drawn from V .

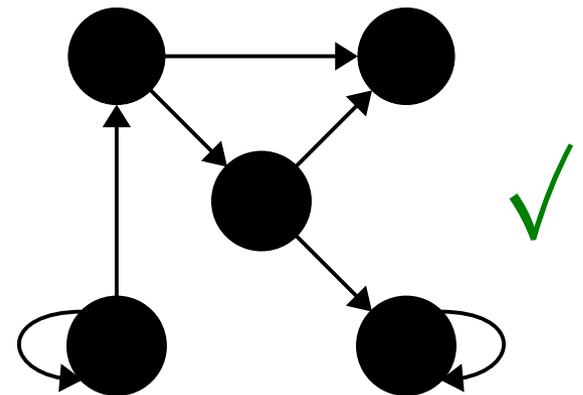
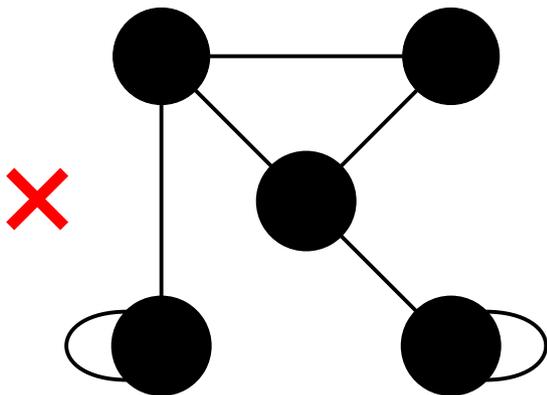
- An **unordered pair** is a set $\{a, b\}$ of two elements $a \neq b$.
- An **undirected graph** is an ordered pair $G = (V, E)$, where
 - V is a set of nodes, which can be anything, and
 - E is a set of edges, which are unordered pairs of nodes drawn from V .



How many of these drawings are of valid undirected graphs?

Self-Loops

- An edge from a node to itself is called a ***self-loop***.
- In (undirected) graphs, self-loops are generally not allowed.
 - Can you see how this follows from the definition?
- In digraphs, self-loops are generally allowed unless specified otherwise.



Time-Out for Announcements!

PS2 Solutions Released

- Solutions to Problem Set Two are now available on the course website.
 - We generally don't release solutions to autograded problems.
 - If you have any questions about those, ping us privately over EdStem or come talk to us at our office hours.
- PS3 is due this Friday at 2:30PM. Ask questions if you have them! That's what we're here for.

Office Hours Logistics

- Thanks for coming to office hours - it's great that you're getting help when you need it!
- Two requests from the TAs to improve their efficiency and let them help more people:
 - ***Come to us when it's your turn.*** The Huang Basement is a big area and it's hard to find you when it's your turn to be helped. Instead, we'll stay in a well-marked area, and please come over to us so we can help you.
 - ***Zoom is limited to special circumstances.*** If you are an SCPD student or in COVID isolation, please feel free to call in over Zoom. Otherwise, please physically be present at office hours.
- Thanks all!

Midterm Elections

- Midterm elections are Tuesday, November 8th.
- Are you a US citizen, eligible to vote, and interested in registering to vote in Santa Clara County? You can take a voter registration form.
- If you are registered to vote with your Stanford address, here's some quick facts:
 - You can look up your polling place using [this official website](#). You can also [vote by mail](#).
 - Your Congresswoman is Anna Eshoo. As usual, all House seats are contested this year.
 - Your US Senators are Dianne Feinstein and Alex Padilla. Alex Padilla's seat is being contested in the upcoming election.
 - Your California State Assemblyman is Marc Berman. As usual, all California State Assembly seats are up for election this year.
 - Your California State Senator is Josh Becker. His seat is not up for election this year.
- All this information is correct to the best of my knowledge; *please correct me if it contains mistakes!*

Midterm Exam Logistics

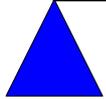
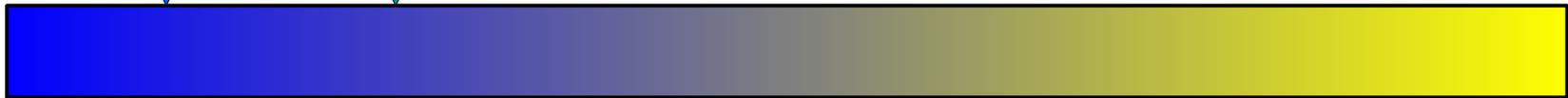
- Our first midterm exam is next **Monday, October 24st**, from **7:00PM - 10:00PM**. Locations are divvied up by last (family) name:
 - A - L: Go to STLC 111.
 - M - Z: Go to Bishop Auditorium.
- You're responsible for Lectures 00 - 05 and topics covered in PS1 - PS2. Later lectures (functions forward) and problem sets (PS3 onward) won't be tested here. Exam problems may build on the written or coding components from the problem sets.
- The exam is closed-book, closed-computer, and limited-note. You can bring a double-sided, 8.5" × 11" sheet of notes with you to the exam, decorated however you'd like.
- After we grade the midterm, you will have a chance to revise and resubmit your answers. Your new score will be the max of your old score and 85% of your new score. More details later.
- We have sent out emails to all students who have requested an alternate exam time. If you can't take the exam during this time or you have OAE accommodations and haven't heard from us, contact us **immediately** so we can book rooms.

Midterm Exam

- ***We want you to do well on this exam.***
 - We're not trying to “weed out” weak students.
 - We're not trying to enforce a curve where there isn't one.
 - We want you to show what you've learned up to this point so that you get a sense for where you stand and where you can improve.
- The purpose of this midterm is to give you a chance to show what you've learned in the past few weeks.
- It is not designed to assess your “mathematical potential” or “innate mathematical ability.”

Preparing for the Exam

CS106A *CS103*



*Dance
Class*

*Philosophy
Class*

Learn by doing.

Learn by reading.

Extra Practice Problems

- Up on the course website, you'll find Extra Practice Problems 1, a set of 24 practice problems on the topics covered by the upcoming midterm.
- Many of these are old midterm questions. Some are just really fun problems we thought you might enjoy working through.
- Take the time to work through some of these problems. This is, perhaps, the best way to study.

You can always run your code and just see what happens!

Checking a proof requires human expertise.

CS106A

CS103

Learning to Speak

Building a Rocket

Rapid iteration.
Constant, small feedback.

Slower iteration.
Infrequent, large feedback.



Doing Practice Problems

- As you work through practice problems, ***keep other humans in the loop!***
- Ask your problem set partner to review your answers and offer feedback – and volunteer to do the same!
- Post your answers as private questions on EdStem and ask for TA feedback!
- ***Feedback loops are key to improving!***

Preparing for the Exam

- We've posted a "Preparing for the Exam" page on the course website with full details and logistics.
- It also includes advice from former CS103 students about how to do well here.
- Check it out - there are tons of goodies there!

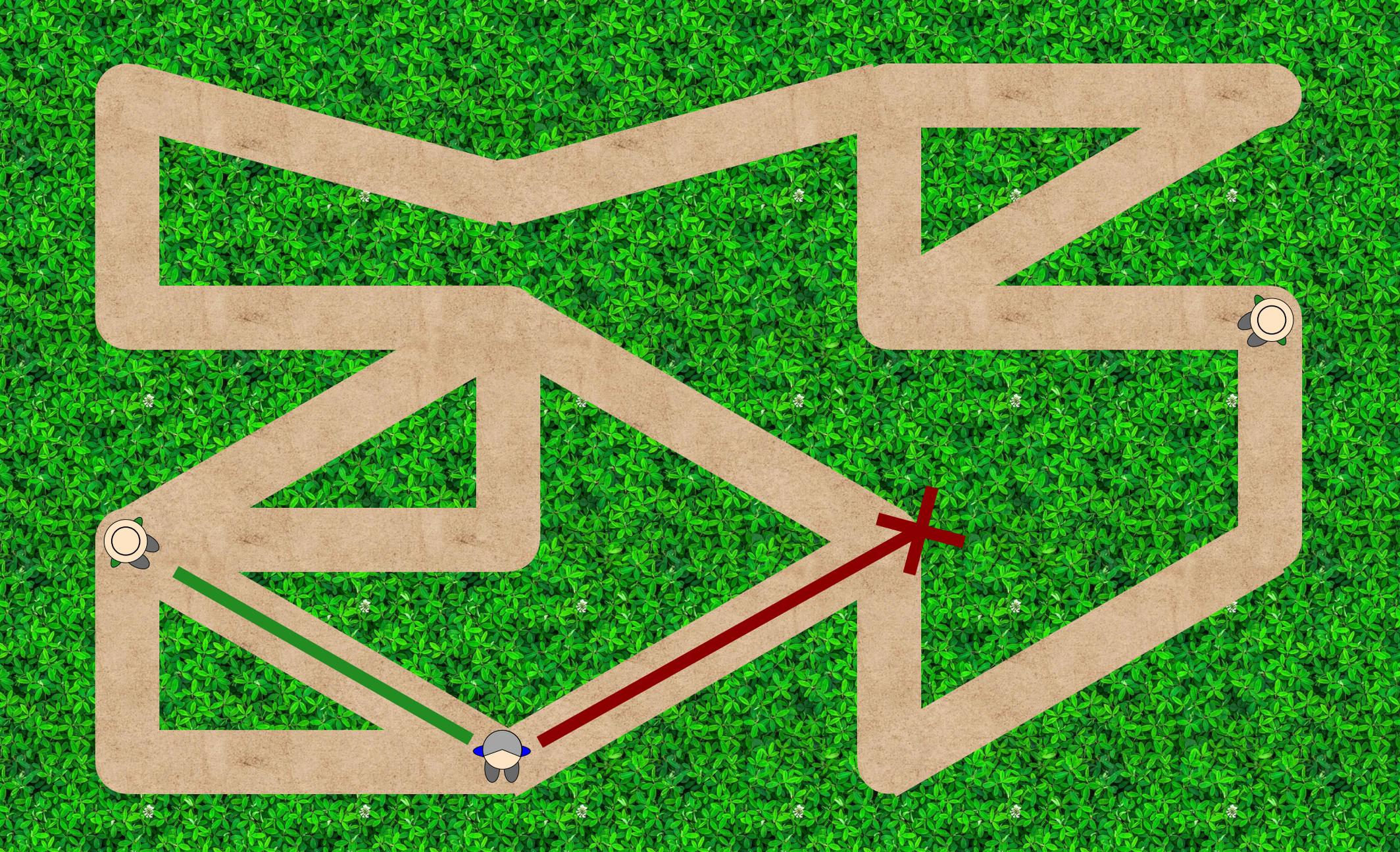
~~Your Questions~~

Next time!

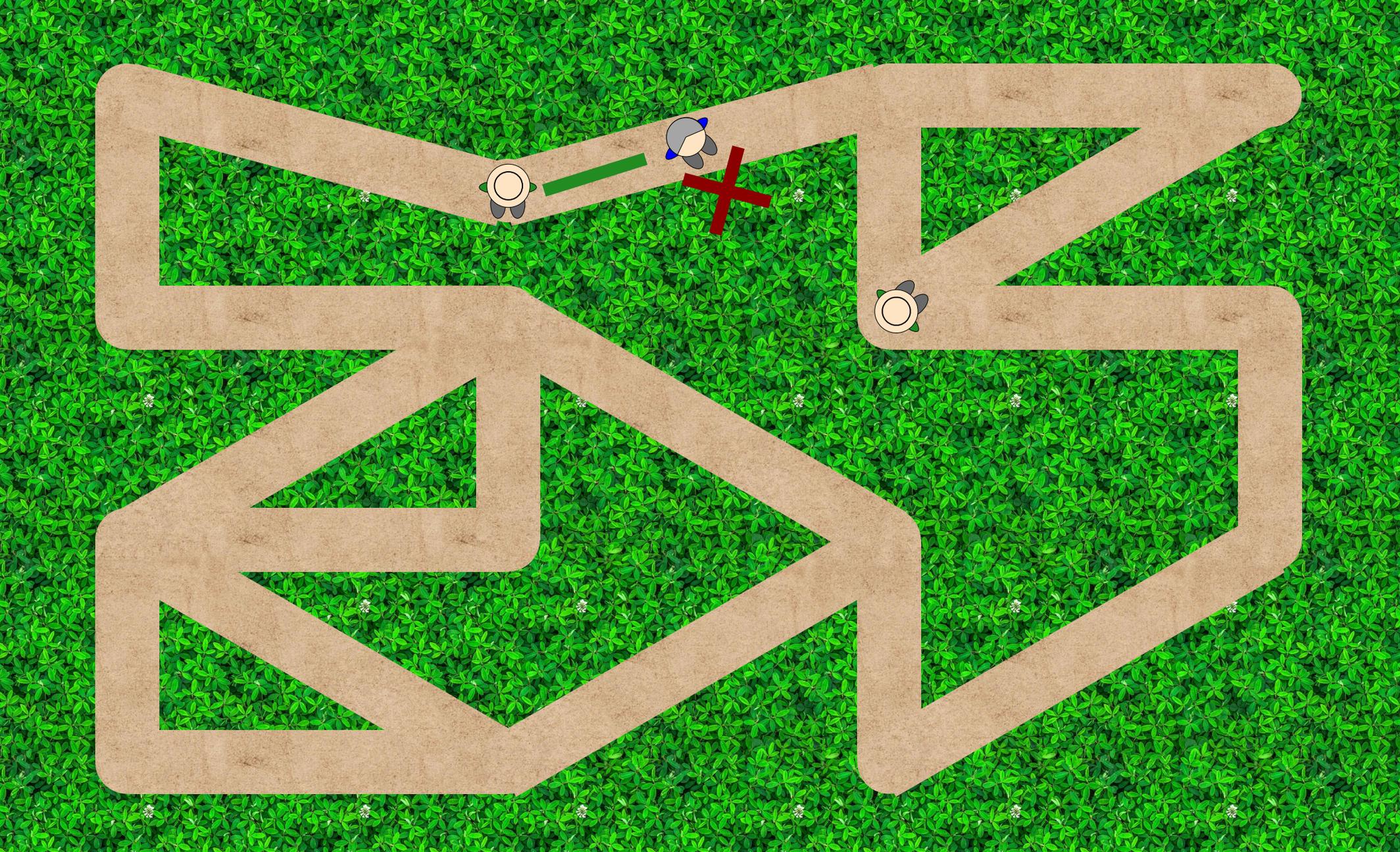
Back to CS103!

Independent Sets and Vertex Covers

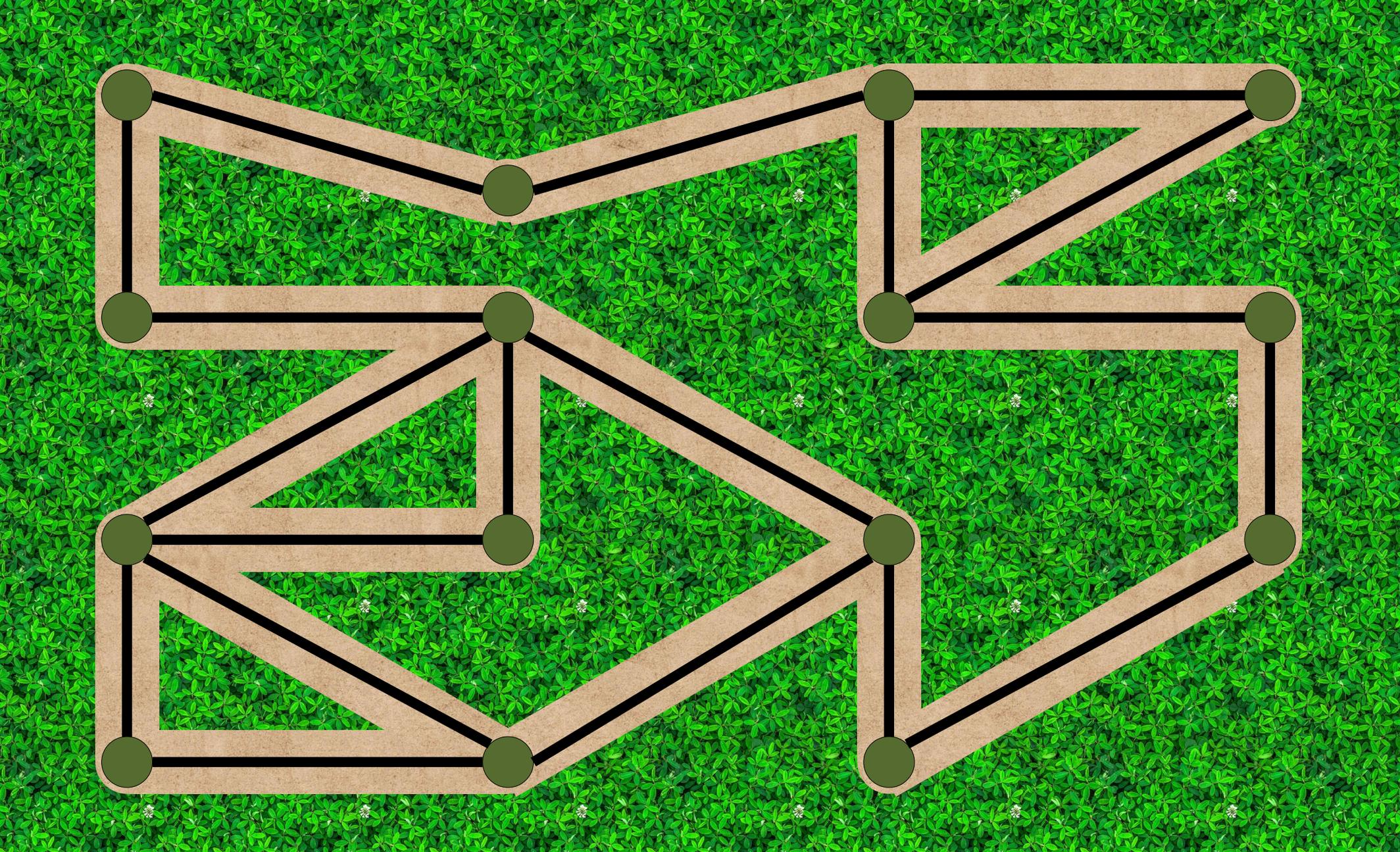
Two Motivating Problems



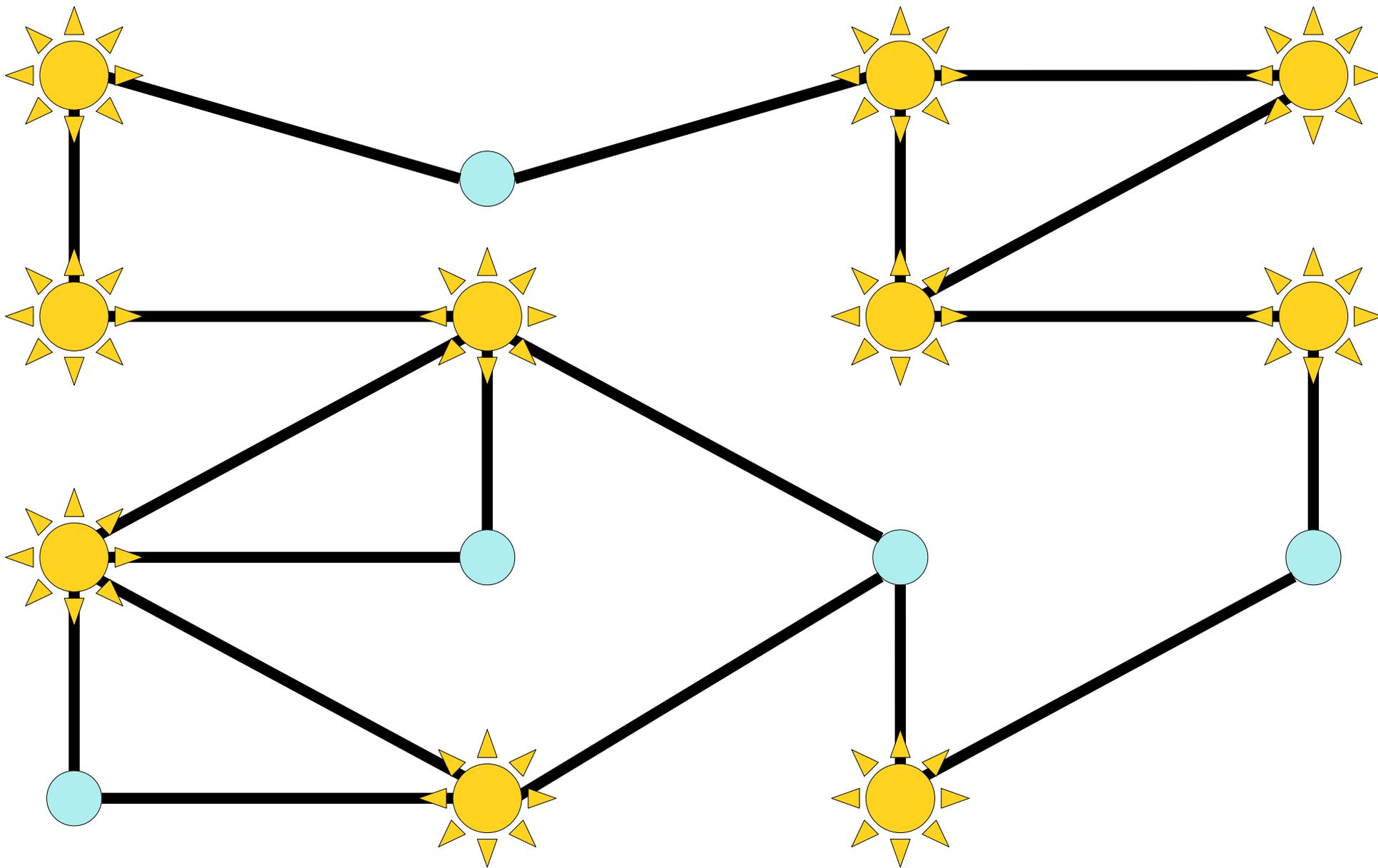
Place park rangers in these forest trails so that a hiker anywhere on a trail can see a park ranger.



Place park rangers in these forest trails so that a hiker anywhere on a trail can see a park ranger.



Place park rangers in these forest trails so that a hiker anywhere on a trail can see a park ranger.



Choose at least one endpoint of each edge.

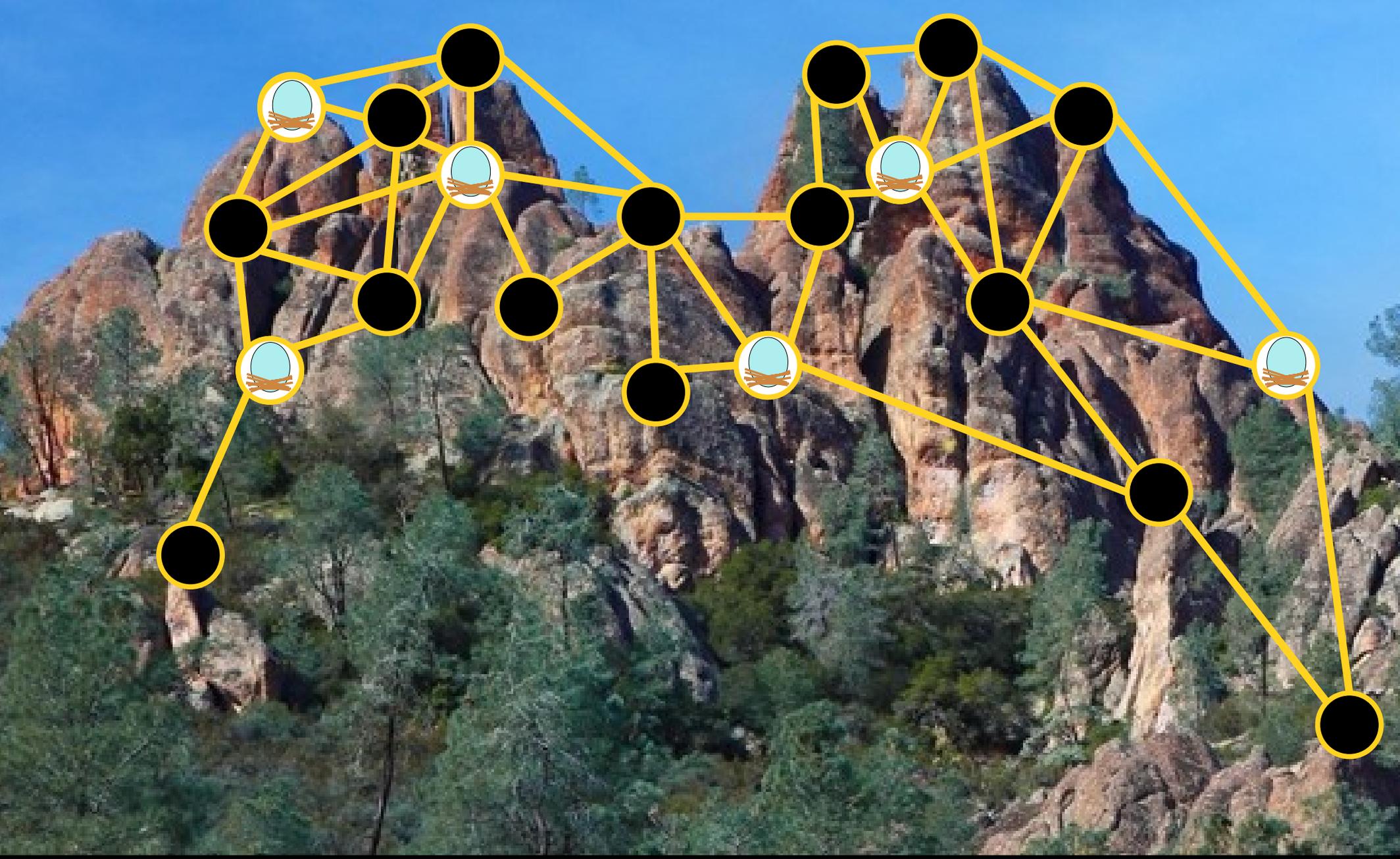
Vertex Covers

- Let $G = (V, E)$ be an undirected graph. A **vertex cover** of G is a set $C \subseteq V$ such that the following statement is true:

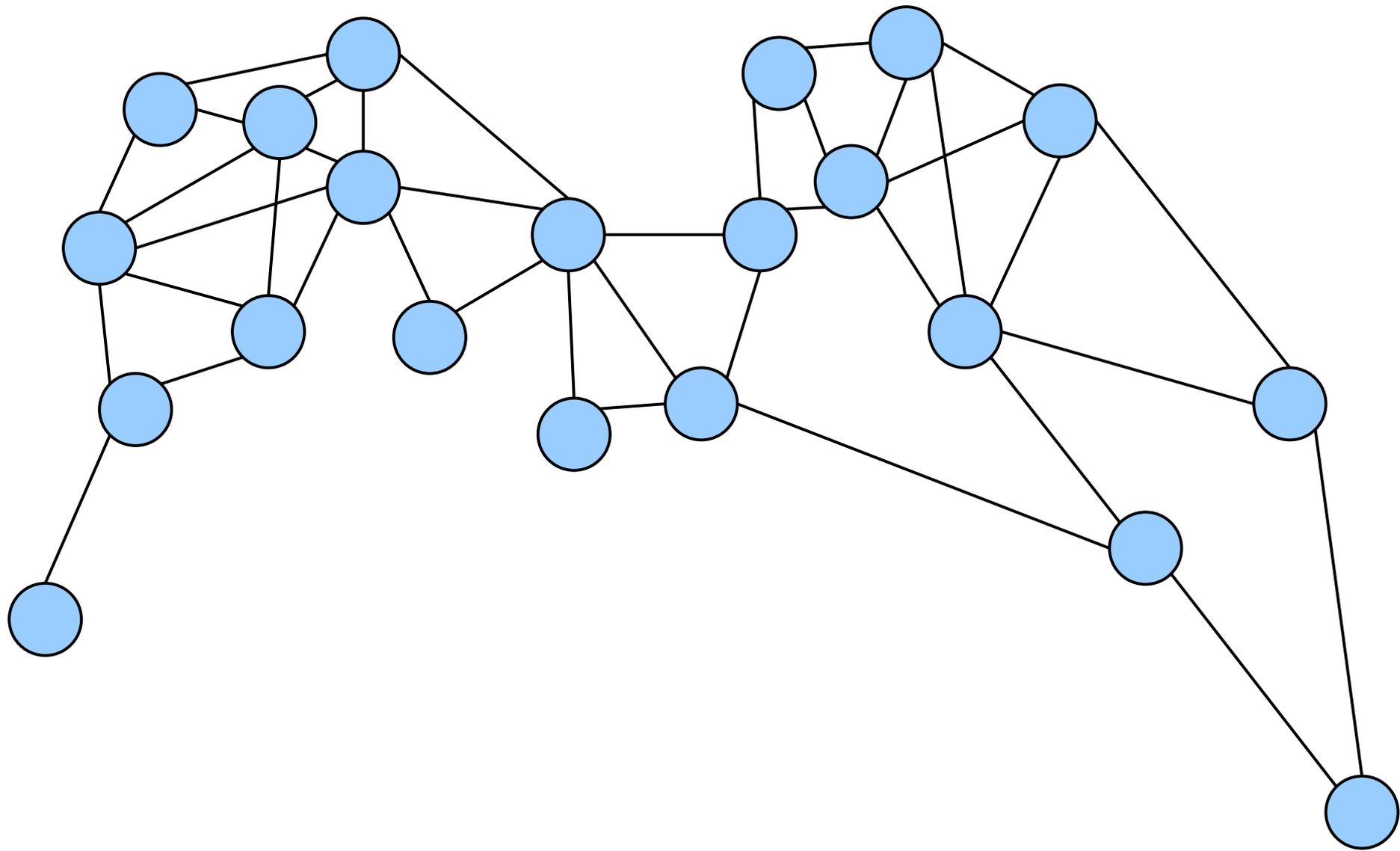
$$\forall x \in V. \forall y \in V. (\{x, y\} \in E \rightarrow (x \in C \vee y \in C))$$

(“Every edge has at least one endpoint in C .”)

- Intuitively speaking, a vertex cover is a set formed by picking at least one endpoint of each edge in the graph.
- Vertex covers have applications to placing streetlights/benches/security guards, as well as in gene sequencing, machine learning, and combinatorics.



Set up nests for the California condor. Condors are territorial and won't nest if they can see other condors.



Choose a set of nodes, no two of which are adjacent.

Independent Sets

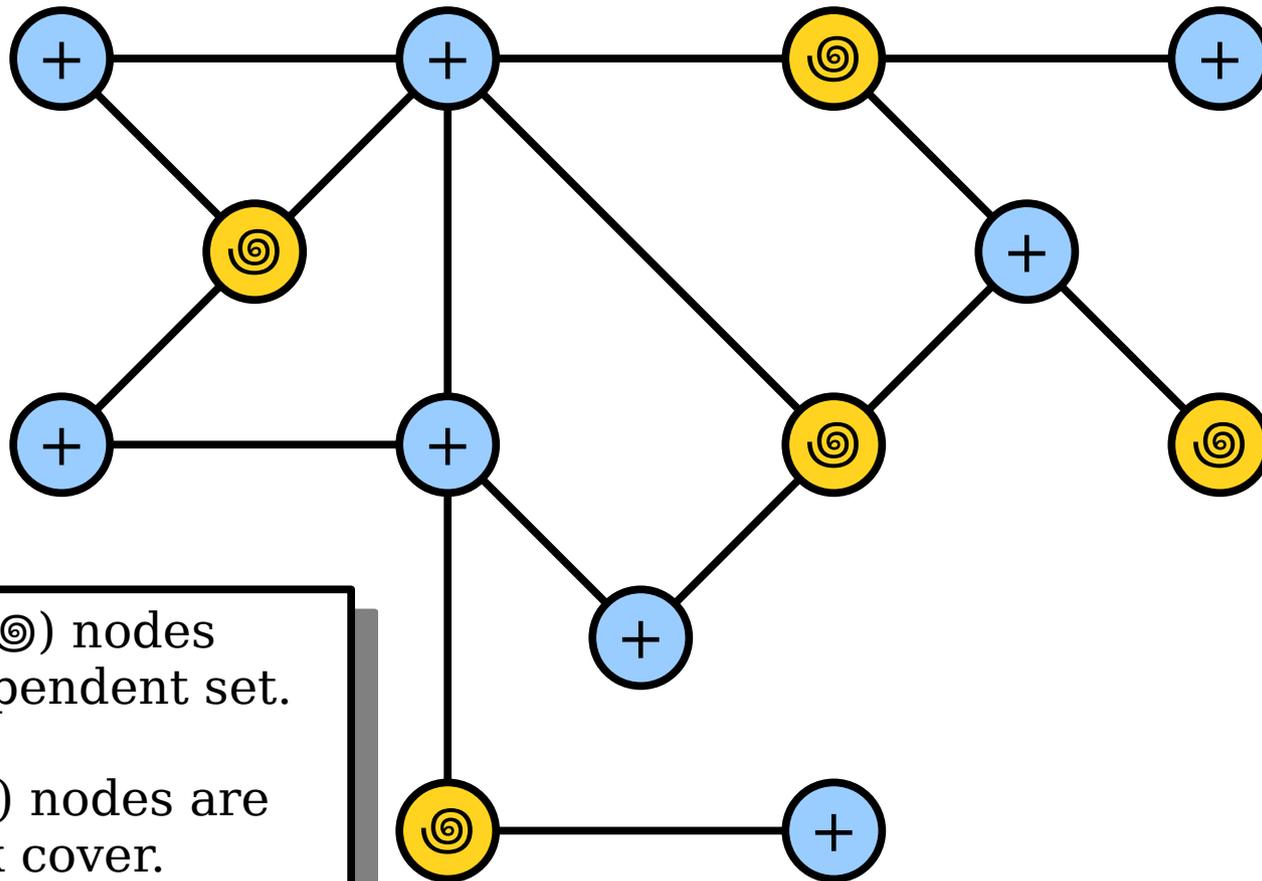
- If $G = (V, E)$ is an (undirected) graph, then an ***independent set*** in G is a set $I \subseteq V$ such that

$$\forall u \in I. \forall v \in I. \{u, v\} \notin E.$$

(“No two nodes in I are adjacent.”)

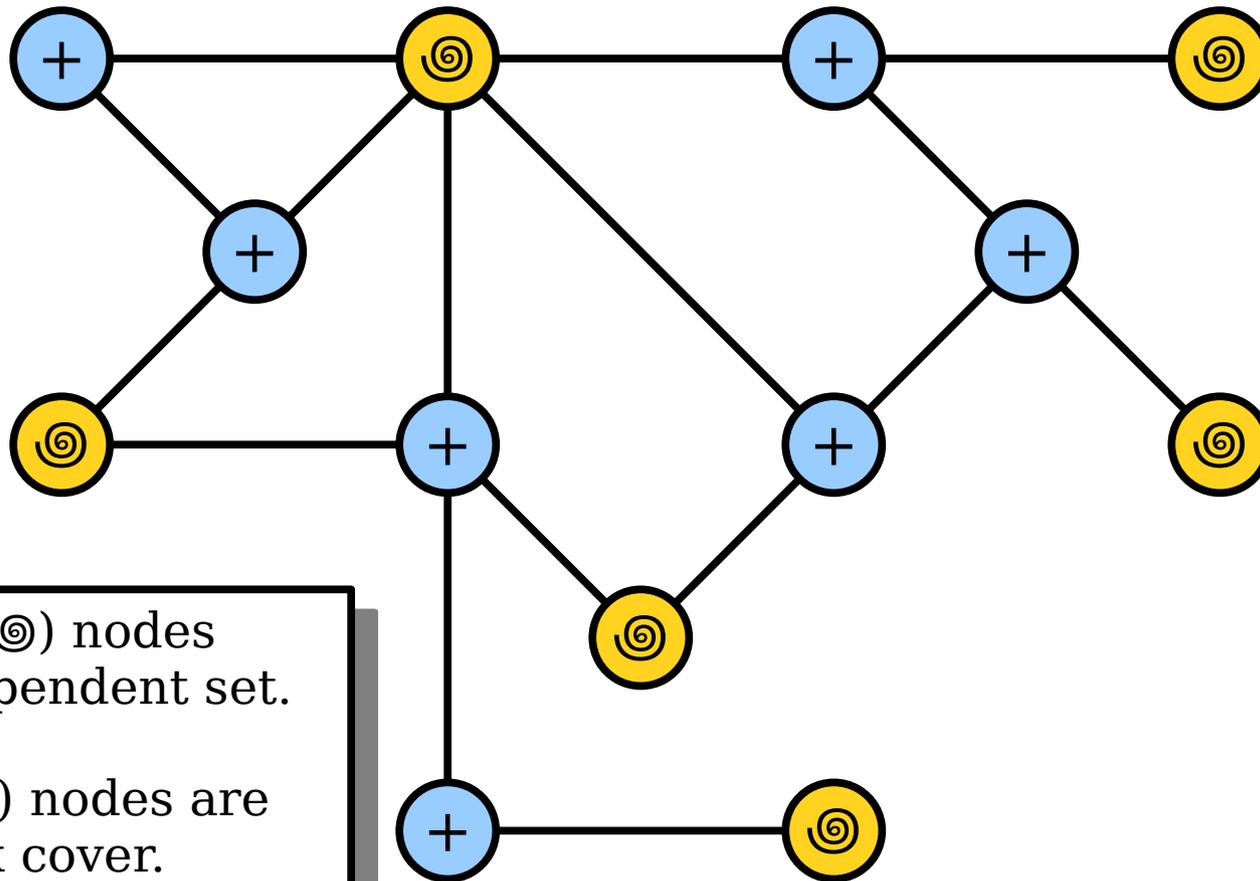
- Independent sets have applications to resource optimization, conflict minimization, error-correcting codes, cryptography, and more.

A Connection



■ The spiral (⊙) nodes are an independent set.
■ The plus (+) nodes are a vertex cover.

Independent sets and vertex covers are related.



■ The spiral (⊙) nodes are an independent set.

■ The plus (+) nodes are a vertex cover.

Theorem: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. Then C is a vertex cover of G if and only if $V - C$ is an independent set in G .

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G , then $V - C$ is an independent set in G .

What We're Assuming

G is a graph.

C is a vertex cover of G .

$$\forall u \in V. \forall v \in V. (\{u, v\} \in E \rightarrow$$
$$u \in C \vee v \in C$$
$$)$$

We're assuming a universally-quantified statement. That means we *don't do anything right now* and instead wait for an edge to present itself.

What We Need To Show

$V - C$ is an independent set in G .

$$\forall x \in V - C.$$
$$\forall y \in V - C.$$
$$\{x, y\} \notin E.$$

We need to prove a universally-quantified statement. We'll ask the reader to pick arbitrary choices of x and y for us to work with.

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G , then $V - C$ is an independent set in G .

What We're Assuming

G is a graph.

C is a vertex cover of G .

$$\forall u \in V. \forall v \in V. (\{u, v\} \in E \rightarrow u \in C \vee v \in C)$$

$x \in V - C$.

$y \in V - C$.

What We Need To Show

$V - C$ is an independent set in G .

$$\forall x \in V - C.$$

$$\forall y \in V - C.$$

$$\{x, y\} \notin E.$$

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G , then $V - C$ is an independent set in G .

What We're Assuming

G is a graph.

C is a vertex cover of G .

$$\forall u \in V. \forall v \in V. (\{u, v\} \in E \rightarrow \\ u \in C \vee v \in C \\)$$

$x \in V$ and $x \notin C$.

$y \in V$ and $y \notin C$.

What We Need To Show

$V - C$ is an independent set in G .

$$\forall x \in V - C.$$

$$\forall y \in V - C.$$

$$\{x, y\} \notin E.$$

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G , then $V - C$ is an independent set in G .

What We're Assuming

G is a graph.

C is a vertex cover of G .

$$\forall u \in V. \forall v \in V. (\{u, v\} \in E \rightarrow u \in C \vee v \in C)$$

$x \in V$ and $x \notin C$.

$y \in V$ and $y \notin C$.

What We Need To Show

$V - C$ is an independent set in G .

$$\forall x \in V - C.$$

$$\forall y \in V - C.$$

$$\{x, y\} \notin E.$$



If this edge exists,
at least one of x
and y is in C .

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G , then $V - C$ is an independent set of G .

Proof: Assume C is a vertex cover of G . We need to show that $V - C$ is an independent set of G . To do so, pick any nodes $x, y \in V - C$; we will show that $\{x, y\} \notin E$.

Suppose for the sake of contradiction that $\{x, y\} \in E$. Because $x, y \in V - C$, we know that $x \notin C$ and $y \notin C$. However, since C is a vertex cover of G and $\{x, y\} \in E$, we also see that $x \in C$ or $y \in C$, contradicting our previous statement.

We've reached a contradiction, so our assumption was wrong. Therefore, we have $\{x, y\} \notin E$, as required. ■

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G , then $V - C$ is not an independent set in G .

How do we express the following statements in FOL?

“ $V - C$ is not an independent set in G .”

“ C is not a vertex cover of G .”

Taking Negations

- What is the negation of this statement, which says “ $V - C$ is an independent set?”

$$\forall u \in V - C. \forall v \in V - C. \{u, v\} \notin E$$

Taking Negations

- What is the negation of this statement, which says “ $V - C$ is an independent set?”

$$\exists u \in V - C. \exists v \in V - C. \{u, v\} \in E$$

- This says “there are two adjacent nodes in $V - C$.”

Taking Negations

- What is the negation of this statement, which says “ C is a vertex cover?”

$$\forall u \in C. \forall v \in C. (\{u, v\} \in C \rightarrow \\ u \in C \quad v \quad v \in C \\)$$

Taking Negations

- What is the negation of this statement, which says “ C is a vertex cover?”

$$\exists u \in C. \exists v \in C. (\{u, v\} \in C \wedge \\ u \notin C \quad \wedge \quad v \notin C \\)$$

- This says “there is an edge where both endpoints aren’t in C .”

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G , then $V - C$ is not an independent set in G .

What We're Assuming

G is a graph.

C is not a vertex cover of G .

$$\exists u \in V. \exists v \in V. (\{u, v\} \in E \wedge u \notin C \wedge v \notin C)$$

We're assuming an existentially-quantified statement, so we'll *immediately* introduce variables u and v .

What We Need To Show

$V - C$ is not an ind. set in G .

$$\exists x \in V - C.$$
$$\exists y \in V - C.$$
$$\{x, y\} \in E.$$

We're proving an existentially-quantified statement, so we *don't* introduce variables x and y . We're on a scavenger hunt!

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G , then $V - C$ is not an independent set in G .

What We're Assuming

G is a graph.

C is not a vertex cover of G .

$$u \in V - C.$$

$$v \in V - C.$$

$$\{u, v\} \in E.$$

What We Need To Show

$V - C$ is not an ind. set in G .

$$\exists x \in V - C.$$

$$\exists y \in V - C.$$

$$\{x, y\} \in E.$$

Any ideas about what we should pick x and y to be?

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G , then $V - C$ is not an independent set of G .

Proof: Assume C is not a vertex cover of G . We need to show that $V - C$ is not an independent set of G .

Since C is not a vertex cover of G , we know that there exists nodes $x, y \in V$ where $\{x, y\} \in E$, where $x \notin C$, and where $y \notin C$. Because $x \in V$ and $x \notin C$, we know that $x \in V - C$. Similarly, we see that $y \in V - C$.

This means that $\{x, y\} \in E$, that $x \in V - C$, and that $y \in V - C$, and therefore that $V - C$ is not an independent set of G , as required. ■

Finding an IS or VC

- The previous theorem means that finding a large IS in a graph is equivalent to finding a small VC.
 - If you've found one, you've found the other!
- **Open Problem:** Design an algorithm that, given an n -node graph, finds either the largest IS or smallest VC “efficiently,” where “efficiently” means “in time $O(n^k)$ for some $k \in \mathbb{N}$.”
 - There's a \$1,000,000 bounty on this problem - we'll see why in Week 10.

Recap for Today

- A **graph** is a structure for representing items that may be linked together. **Digraphs** represent that same idea, but with a directionality on the links.
- Graphs can't have **self-loops**; digraphs can.
- **Vertex covers** and **independent sets** are useful tools for modeling problems with graphs.
- The complement of a vertex cover is an independent set, and vice-versa.

Next Time

- ***Paths and Trails***
 - Walking from one point to another.
- ***Indegrees and Outdegrees***
 - Counting how many neighbors you have, in the directed case.
- ***Teleporting a Train***
 - Can you get stuck in a loop?